Week 8 Orthogonal Projection

Def Let X be a vector space $Y, Z \leq X$ be subspace. We say that X is a direct sum of Y and Z denoted by $X = Y \oplus Z$ if for any $x \in X$, $\exists$ unique $y \in Y$, $z \in Z$ such that

$$x = y + z$$

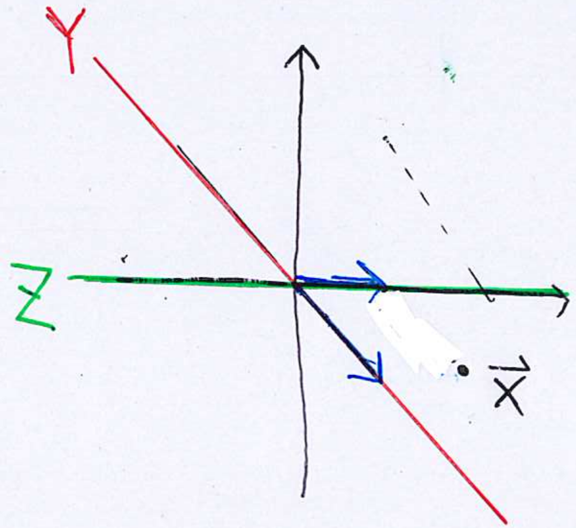
Rmk If $X = Y \oplus Z$, then

① $Y \cap Z = \{0\}$

② $\dim X = \dim Y + \dim Z$

eg $X = \mathbb{R}^2$ $Y = \{(x,y) \in \mathbb{R}^2 : x + y = 0\}$

$Z = \{(x,0) \in \mathbb{R}^2 : x \in \mathbb{R}\}$

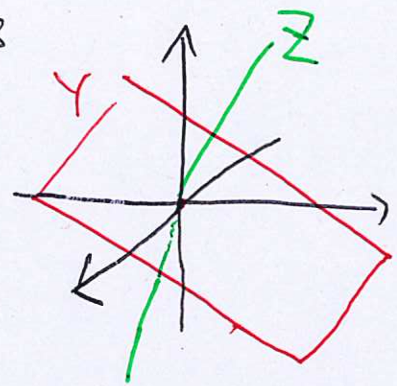


Let $x = (2,-1)$

$$x = \underset{Y}{(1,-1)} + \underset{Z}{(1,0)}$$

$\mathbb{R}^2 = Y \oplus Z$

$\mathbb{R}^3$



$\mathbb{R}^3 = Y \oplus Z$

$Z \cap Y = \{\vec{0}\}$

<u>Defn</u> If X is an inner product space

$S \subset X$ be a subset.

Define the orthogonal complement of S

to be $$S^{\perp} = \{x \in X : \langle x, y \rangle = 0 \ \forall y \in S\}$$

<u>Prop</u> $S^{\perp}$ is a subspace of X

<u>Pf</u> ① $\langle \vec{0}, y \rangle = 0$ for any $y \in S \Rightarrow \vec{0} \in S^{\perp}$

② If $x_1, x_2 \in S^{\perp}$, then for any $y \in S$

$$\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$$
$$= 0 + 0 = 0$$
$$\Rightarrow x_1 + x_2 \in S^{\perp}$$

③ Similarly, if $x \in S^{\perp}$, $\alpha \in \mathbb{F}$, then

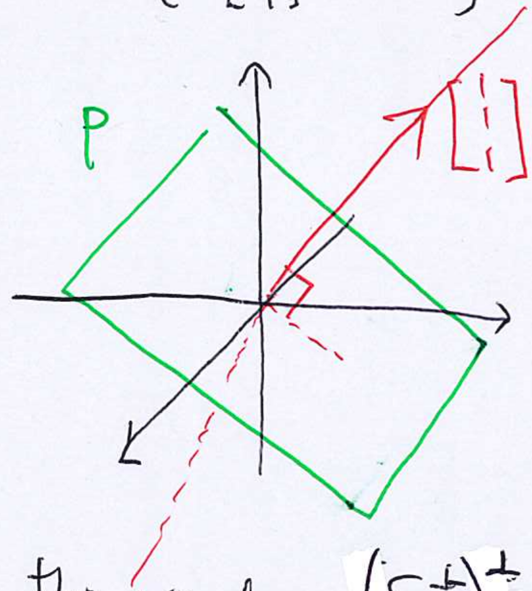
$$\langle \alpha x, y \rangle = \alpha \langle x, y \rangle = 0 \quad \forall y \Rightarrow \alpha x \in S^{\perp}$$

<u>eg</u>

$S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$ is a basis of

$P: x + y + z = 0$

$$S^{\perp} = \left\{ \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} : \alpha \in \mathbb{R} \right\}$$



In this example, $(S^{\perp})^{\perp} = P$

<u>eg</u>

$\{0\}^{\perp} = X \qquad X^{\perp} = \{0\}$

Last time :

Inner Product Space $\rightsquigarrow$ Normed Space $\rightsquigarrow$ Metric Space

(Continuity
Convergence)

Def A Hilbert space is a complete inner product space

eg (1) Closed subspace of Hilbert space

(2) Finite dimensional inner product space

(3) ℓ^2, $L^2([a,b])$

Lemma 3.2-2 Inner product is continuous :

If $x_n \to x$, $y_n \to y$, then $\langle x_n, y_n\rangle \to \langle x, y\rangle$

Pf

$$
\begin{aligned}
&|\langle x_n, y_n\rangle - \langle x, y\rangle| \\
&= |\langle x_n, y_n\rangle - \langle x, y_n\rangle + \langle x, y_n\rangle - \langle x, y\rangle| \\
&= |\langle x_n - x, y_n\rangle + \langle x, y_n - y\rangle| \\
&\leq |\langle x_n - x, y_n\rangle| + |\langle x, y_n - y\rangle| \\
&\leq \|x_n - x\|\,\|y_n\| + \|x\|\,\|y_n - y\|
\end{aligned}
$$

$\lim_{n\to\infty} \text{RHS} = 0 \cdot \|y\| + \|x\|\, 0 = 0$

Moreover $\quad \text{LHS} \geq 0$

Sandwich Thm $\Rightarrow \lim_{n\to\infty} |\langle x_n, y_n\rangle - \langle x, y\rangle| = 0$

$\Rightarrow \langle x_n, y_n\rangle \to \langle x, y\rangle$

Distance function

Let X be a metric space

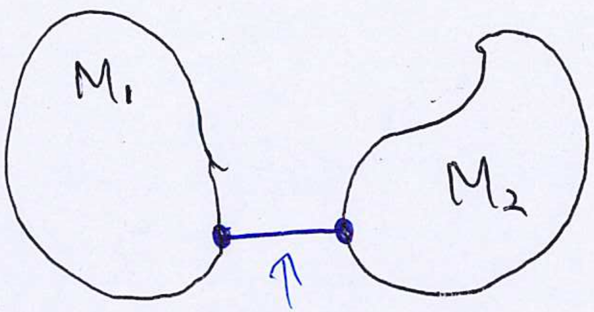
$M_1, M_2 \subset X$ be subset, $x \in X$

Define

$$d(x, M_1) = \inf_{a \in M_1} d(x, a)$$

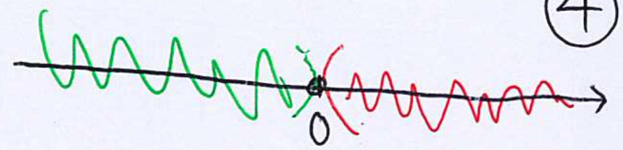
$$d(M_1, M_2) = \inf_{a \in M_1} d(a, M_2)$$

$$= \inf_{\substack{a \in M_1 \\ b \in M_2}} d(a, b)$$



$d(M_1, M_2)$

eg $d(\mathbb{R}^-, \mathbb{R}^+) = 0$



Rmk

$$x \in A \underset{\cancel{\Leftarrow}}{\Rightarrow} d(x, A) = 0$$

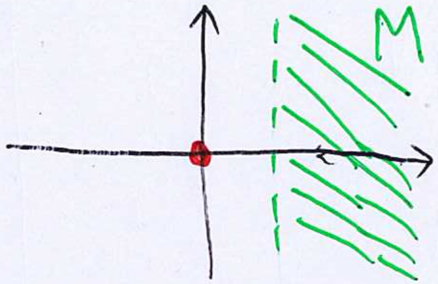
$$A \cap B \neq \phi \underset{\cancel{\Leftarrow}}{\Rightarrow} d(A, B) = 0$$

Q Given x and M, does there exist $y \in M$ s.t. $d(x, M) = d(x, y)$? If so, is it unique?

Picture $\mathbb{R}^2$

$M = \{(x,y) : x > 1\}$

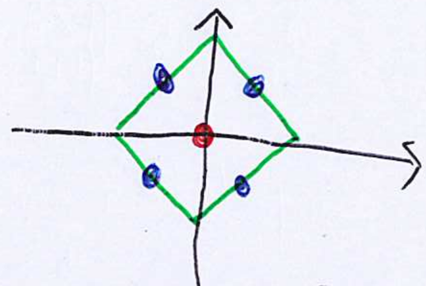
$x = \{(0,0)\}$



Such a y DNE

$M = \{(x,y) : |x| + |y| = 1\}$

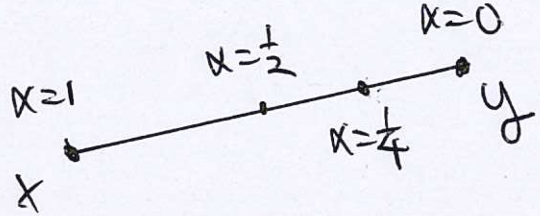
$x = \{(0,0)\}$



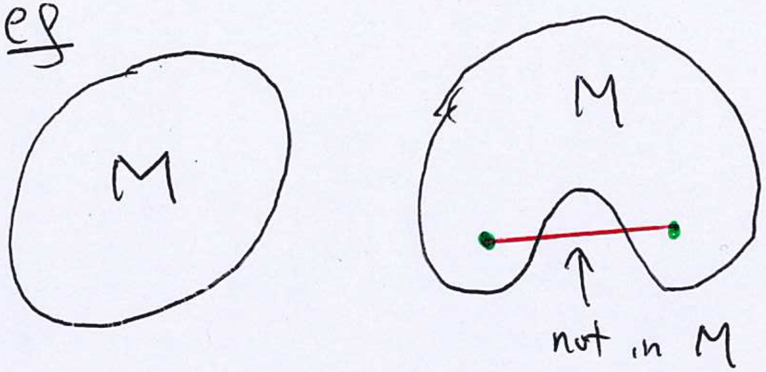
4 such y : $\left(\pm\frac{1}{2}, \pm\frac{1}{2}\right)$

Defn Let X be a vector space. A subset $M \subset X$ is said to be *Convex* if for any $x, y \in M$, $\alpha \in \mathbb{R}$ with $0 \leq \alpha \leq 1$, then

$$\alpha x + (1-\alpha) y \in M$$



eg

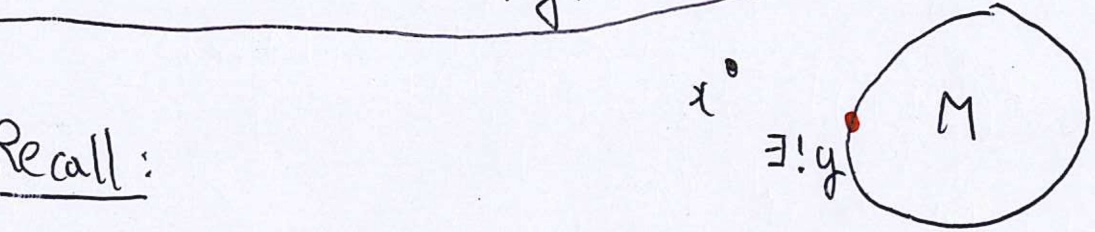


Convex Not convex

Thm 3.3-1 (Minimizing vector)

Let X be an inner product space, $M \subseteq X$ be an non-empty, convex, complete subset. Then for any $x \in X$, $\exists$ unique $y \in M$ s.t.

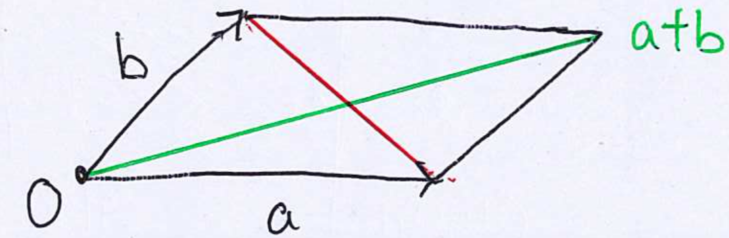
$$d(x, M) = d(x, y)$$



Recall:

Parallelogram equality: $a, b \in X$

$$\|a+b\|^2 + \|a-b\|^2 = 2(\|a\|^2 + \|b\|^2)$$



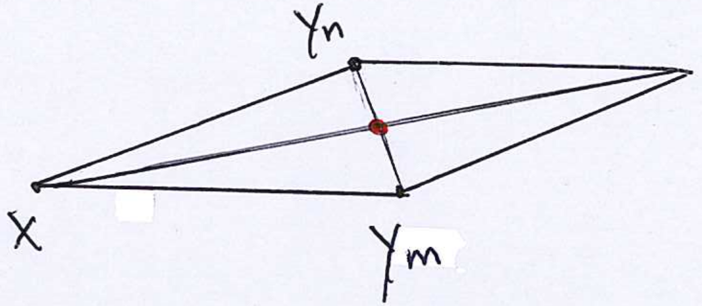
Pf $\boxed{\text{Existence of } y}$

Let $\delta = d(x, M)$

$\exists$ sequence $y_n \in M$ such that

$$\delta_n = d(x, y_n) \longrightarrow \delta$$

Want to show y_n is Cauchy



Let $m \geq n$.

Parallelogram equality $\Rightarrow$

$$2\left(\|y_n - x\|^2 + \|y_m - x\|^2\right)$$

$$= \|(y_n - x) + (y_m - x)\|^2 + \|(y_n - x) - (y_m - x)\|^2$$

$$= 4\left\|\frac{y_n + y_m}{2} - x\right\|^2 + \|y_n - y_m\|^2$$

$\because$ M is convex, $\therefore$ $\dfrac{y_n + y_m}{2} \in M$

$$\Rightarrow \quad 2(\delta_n^2 + \delta_m^2) \geq 4\delta^2 + \|y_n - y_m\|^2$$

$$\Rightarrow \quad \|y_n - y_m\|^2 \leq 2(\delta_n^2 + \delta_m^2) - 4\delta^2$$

When $n \to \infty$ R.H.S $\to$ $2(\delta^2 + \delta^2) - 4\delta^2 = 0$

Sandwich theorem $\Rightarrow$ $\|y_n - y_m\|^2 \to 0$ as $n \to \infty$

$\Rightarrow$ (y_n) is Cauchy

M is complete $\Rightarrow$ (y_n) is convergent, let $y_n \to y \in M$

Then $d(x, y) = d(x, \lim_{n\to\infty} y_n)$

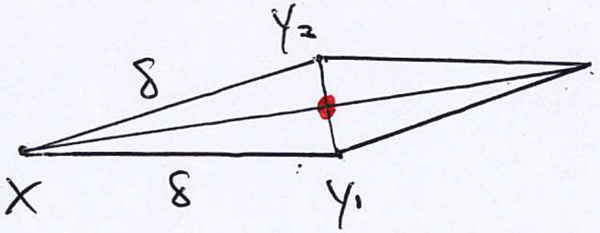
$$= \lim_{n\to\infty} d(x, y_n)$$

$$= \lim_{n\to\infty} \delta_n = \delta \Rightarrow \text{Existence}$$

Uniqueness of y

Suppose $y_1, y_2 \in M$ s.t.

$$d(x,y_1) = d(x,y_2) = \delta$$



Parallogram Equality $\Rightarrow$

$$2(\|y_1-x\|^2+\|y_2-x\|^2)$$

$$= 4\left\|\frac{y_1+y_2}{2}-x\right\|^2 + \|y_1-y_2\|^2$$

$$\Rightarrow 4\delta^2 \geq 4\delta^2 + \|y_1-y_2\|^2$$

$$\Rightarrow \|y_1-y_2\| = 0 \Rightarrow y_1-y_2=0$$

$$\Rightarrow y_1 = y_2 \Rightarrow \text{uniqueness}$$

Rmk for Thm 3.3-1

The assumption that X is an inner product space is important (Parallogram Equality)

eg $X = \ell^\infty = \{\vec{x} = (x_1, x_2, \cdots) : \sup_{i\in\mathbb{N}} |x_i| < \infty\}$

$M = c_0 = \{\vec{y} = (y_1, y_2, \cdots) : \lim_{n\to\infty} y_n = 0\}$

Note that M is covex, complete, non-empty

Take $x = (1,1,1,1,1,\cdots) \in \ell^\infty$

Then $d(x,M) = 1$

$\exists$ infinitely many $y \in M$ such that $d(x,y)=1$

eg. $y = e_1 = (1,0,0,0,\cdots)$

or $e_n = (0,0,0,\cdots 0,1,0,0,\cdots)$

Special Case of Thm 3.3-1

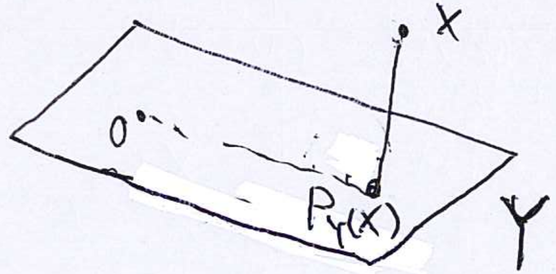
If $M = Y$ is a closed subspace of a Hilbert space H

Ex Show that M is convex and complete

Defn Let H be a Hilbert space, $Y \subseteq H$ is a closed subspace. For any $x \in H$, define $P_Y(x)$ (or $P(x)$) to be the unique vector in Y such that

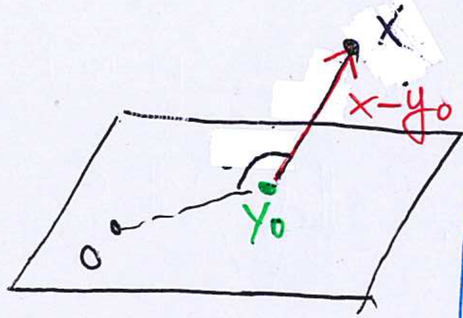
$$d(x, Y) = d(x, P_Y(x)) = \| x - P_Y(x) \|$$

Rmk $P_Y(x)$ is well-defined because of 3.3-1



Lemma 3.3-2 Let H and Y be as in 3.3-1. Let $x \in H$. Then for $y_0 \in Y$

$$\boxed{y_0 = P_Y(x) \iff x - y_0 \in Y^{\perp}}$$



(Not shortest if not right angle)

Because of the lemma, $P_Y(x)$ is called the orthogonal projection of x onto Y

Pf ($\Rightarrow$) Suppose $y_0 = P_Y(x)$

Let $z = x - y_0$

Suppose $z \notin Y^\perp$ (Pf by contradiction)

$\Rightarrow \exists\, y_1 \in Y$ such that $\langle z, y_1 \rangle = \beta \neq 0$

Then $(\therefore y_1 \neq 0)$

$$\|z - \alpha y_1\|^2 = \langle z - \alpha y_1, z - \alpha y_1 \rangle$$

$$= \langle z, z\rangle - \bar{\alpha}\langle z, y_1\rangle - \alpha\langle y_1, z\rangle - \alpha\bar{\alpha}\langle y_1, y_1\rangle$$

$$= \langle z, z\rangle - \bar{\alpha}\langle z, y_1\rangle - \alpha\left(\langle y_1, z\rangle - \bar{\alpha}\langle y_1, y_1\rangle\right)$$

Let $\bar{\alpha} = \dfrac{\langle y_1, z\rangle}{\langle y_1, y_1\rangle}$. Then

$$\|z - \alpha y_1\|^2 = \|z\|^2 - \frac{|\langle y_1, z\rangle|^2}{\|y_1\|^2} = \|z\|^2 - \frac{|\beta|^2}{\|y_1\|^2} < \|z\|^2$$

$$\Rightarrow \|x - (y_0 + \alpha y_1)\|^2 < \|x - y_0\|^2$$

$$= d(x, y_0)^2$$

$$= d(x, Y)^2$$

Contradiction because $y_0 + \alpha y_1 \in Y$

$\therefore\ z = x - y_0 \in Y^\perp$

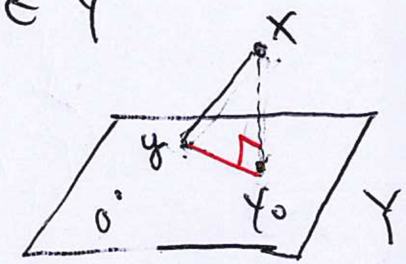
($\Leftarrow$) Suppose $x - y_0 \in Y^\perp$

Let $y \in Y$, then

Y

$\cup$

$y_0 - y \perp x - y_0 \in Y^\perp$

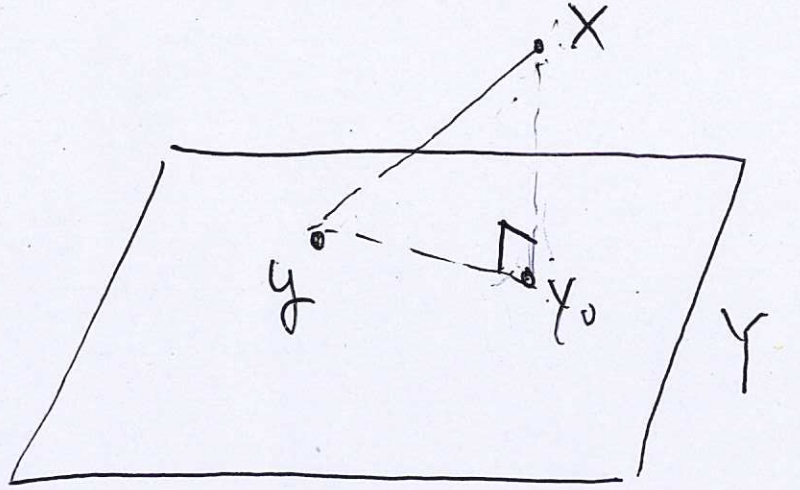


Apply Pythagorean theorem

$$\|x-y\|^2$$
$$= \|(x-y_0)+(y_0-y)\|^2$$
$$= \|x-y_0\|^2 + \|y_0-y\|^2$$
$$\geq \|x-y_0\|^2$$

$\Rightarrow \|x-y\| \geq \|x-y_0\| \quad \forall y \in Y$

$\Rightarrow \quad y_0 = P_Y(x)$



Thm 3.3-4 Let H be a Hilbert Space, $Y \subseteq H$ is a closed subspace of H.
Then $\quad H = Y \oplus Y^{\perp}$

Pf For any $x \in H$, consider $y = P_Y(x) \in Y$

Let $z = x - y = x - P_Y(x)$

Lemma 3.3-2 $\Rightarrow$ $z \in Y^{\perp}$

Also $x = y + z$ with $y \in Y$ and $z \in Y^{\perp}$

For uniqueness

Suppose $y' \in Y$, $z' \in Y^{\perp}$ such that $x = y' + z'$

Then $\quad y + z = y' + z'$

$$y - y' = z' - z$$

$\Rightarrow \langle y-y', y-y' \rangle = \langle \underbrace{y-y'}_{\in Y}, \underbrace{z'-z}_{\in Y^{\perp}} \rangle = 0 \Rightarrow \begin{array}{l} y-y'=0 \\ z'-z=0 \end{array}$

$\Rightarrow y = y', \; z = z'$